Sequences Characterizing k-Trees

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Abstract. A non-decreasing sequence of n integers is the degree sequence of a 1-tree (i.e., an ordinary tree) on n vertices if and only if there are least two 1's in the sequence, and the sum of the elements is $2(n-1)$. We generalize this result in the following ways. First, a natural generalization of this statement is a necessary condition for k-trees, and we show that it is not sufficient for any $k > 1$. Second, we identify non-trivial sufficient conditions for the degree sequences of 2-trees. We also show that these sufficient conditions are *almost* necessary using bounds on the partition function $p(n)$ and probabilistic methods. Third, we generalize the characterization of degrees of 1-trees in an elegant and counter-intuitive way to yield integer sequences that characterize k-trees, for all k.

1 Introduction

1.1 Degree Sequence and Characterization

Definition 1. *The* degree sequence *of an undirected graph $G = (V, E)$ is the list of degrees of its nodes, with duplication, sorted in non-decreasing order. A* graphic sequence *is a sequence of integers which is the degree sequence of a simple undirected graph. That is, a graph that does not contain loops or parallel edges. Graph G* realizes *a degree sequence Δ if Δ is the degree sequence of G.*

The basic sequence recognition problem is to determine whether a sequence of integers is a graphic sequence at all. This problem was solved half a century ago by Havel [9], Hakimi [7] and Erdös and Gallai [3]. Their solutions are constructive. That is, if the sequence is graphic, they show how to construct a simple graph that realizes it.

For a specific graph class $\mathcal{C}$, there can be two types of classification results. The first type is a global classification, where we are given a sequence Δ and need to determine whether *every* simple graph that realizes Δ belongs to $\mathcal{C}$. The second type is an existential classification, where we need to determine whether there *exists* a graph in $\mathcal{C}$ that realizes Δ and, if so, to construct one.

D.Z. Chen and D.T. Lee (Eds.): COCOON 2006, LNCS 4112, pp. 216–225, 2006.

Hammer and Simone [8] studied split graphs, which are graphs that have the property that their node set can be partitioned into a clique and an independent set. Their results imply that if G is a split graph, then any graph with the same degree sequence as G is also a split graph. Furthermore, the degree sequences that are realized by split graphs can be identified in linear time. Another example of a sequence recognition result was conjectured by Erdös et al. [15] and proved by Li et al. [12]. The problem is to find the minimal value $\sigma(k, n)$ such that every graphic sequence of length n without zero terms that sums to $\sigma(k, n)$ can be realized by a graph that contains a clique of size $k + 1$. This value was shown to be $\sigma(k, n) = (k - 1)(2n - k) + 2$. A related result is the Turán number [16] $ex(k, n)$ which is the smallest integer such that a graph with n nodes and $ex(n, k)$ edge is guaranteed to contain a clique of size $k + 1$ [4].

In this paper we consider the characterization problem of k-trees.

1.2 k-Trees and Previous Work

Definition 2. *A* k-tree *is recursively defined as follows.*

1. *A complete graph with $k + 1$ nodes is a k-tree.*
2. *If G is a k-tree and the nodes $v_1, \ldots, v_k$ form a k-clique in G, then the graph obtained by adding a node to G and connecting it by an edge to each of $v_1, \ldots, v_k$ is a k-tree.*

A 1-tree is a tree, hence this definition generalizes the notion of a tree. The minimum degree of a node in a k-tree is k, and in the context of a k-tree, by "leaf" we mean a node of degree k. Given an input graph, it can be determined in time $O(kn)$ whether this graph is a k-tree [5,13]. Every k-tree has treewidth k, and in fact k-trees are instrumental in one of the definitions of treewidth [14]. *Degree sets* of k trees have been studied extensively by Duke and Winkler [1,2,18]. Note that, while degree sequences are ordered in non-decreasing order, the degree set has no sequence information, nor the number of times a certain number may be used as the degree of a vertex. In this sense, characterizing degree sequences is harder than characterizing degree sets. In particular they show that degree sets of 2-trees are indeed characterized by the degree sets of 2-caterpillars, see Definition 6, which are a subclass of 2-trees. In [1], Duke and Winkler show that if D is any finite set of positive integers, which includes 1, then D is the set of vertex degrees (for a slightly different but equivalent definition of "degree") of some k-tree for k=2,3, and 4, and that there is precisely one such set, $D = \{1, 4, 6\}$, which is not the set of degrees of any 5-tree. They also show for each $k \geq 2$ that such a set D is the set of degrees of some k-tree, provided only that D contains some element d, which satisfies $d \geq k(k-1) - 2\left\lfloor \frac{k}{2} \right\rfloor + 3$.

However, prior to our work, degree sequences only of trees were characterized:

Theorem 1 (Folklore). *A degree sequence $\Delta = < d_1, d_2, \ldots, d_n >$ can be realized by a tree iff:*

1. $1 \leq d_i \leq n - 1$ *for all* $1 \leq i \leq n$.
2. $\sum_{i=1}^{n} d_i = 2n - 2$.

1.3 Our Work and Results

This work follows from an effort to characterize degree sequences of 2-trees. Theorem 1 shows that the necessary conditions on the degree sequence of a tree are indeed sufficient. A natural generalization of this theorem would be that, for all $k \geq 0$, the necessary conditions for the degree sequence of a k-tree, see Definition 3, are sufficient. However, in this paper (see Section 2, we show that the conjecture is false for all $k \geq 2$. Following this, in Section 3, we identify the *right* generalization of degree sequences in a way that helps to characterize such sequences that correspond to k-trees. The generalization lies in viewing the degree sequence of a graph in a slightly different way; *the entries of a degree sequence count the number of* 2*-cliques(edges) that contain a 1-clique(a vertex).*

While we show that plausible k-sequences (see Definition 3) do not characterize k-trees, we present some fundamental results on them for $k = 2$. In Section 4 we show that if a plausible 2-sequence contains a 3, then it is the degree sequence of 2-tree. In this proof, we identify a structure of a 2-tree that makes it possible to output such a tree in linear time. Having shown that a plausible 2-sequence which contains a 3 is the degree sequence of a 2-tree, we show in Section 5 that almost every plausible 2-sequence contains a 3 and hence almost every plausible 2-sequence is realizable. This proof is based on the idea that for a certain number n, each plausible 2-sequence corresponds to a partition of $2n - 7$. We then use bounds on the partition function $p(n)$ [10,11,17], the integer function that counts the number of partition of n, to prove the claim.

Throughout the paper, the symbol n usually denotes the size of a k-tree or a degree sequence. We sometimes identify nodes by their degree. For example, by "adding a 3", we mean "adding a node of degree 3".

2 Non-realizable k-Sequences

For 1-trees it turns out that the necessary conditions on the degree sequence are indeed sufficient. The natural conjecture would be that the same holds for k-trees too, for $k \geq 2$. In Lemma 1 we show that this conjecture is false for k-trees by exhibiting one class of sequences that satisfy the necessary conditions but are not realizable by k-trees. To show this we define *plausible k-sequences* as those that satisfy the necessary conditions.

Definition 3. *A sequence of integers $\Delta =< d_1, d_2, \ldots, d_n >$ is a* plausible k-sequence *if the following conditions hold:*

1. $d_i \leq d_{i+1}$ *for all* $1 \leq i < n$.
2. $d_n \leq n - 1$.
3. $d_1 = d_2 = k$.
4. $\sum_{i=1}^{n} d_i = k(2n - k - 1)$.

Lemma 1. *For every $k > 1$, for every integer n such that $b = \frac{k(n+1)}{k+2}$ is a positive integer, the plausible k-sequence $d_1 = d_2 = \ldots = d_{n-k-2} = k, d_{n-k-1} = \ldots = d_n = b$ is not the degree sequence of any k-tree.*

Proof. Consider a k-tree T corresponding to the said plausible k-sequence. Let $L \subseteq T$ be the set of all nodes of degree k. Now $T - L$ induces a k-tree on $k + 2$ nodes, which has two non-adjacent nodes, say a and b, of degree k. Now, no matter in what order we add the vertices of L to obtain the k-tree T from the k-tree $T-L$, we will never be able to *equalize* the degrees of $T-L$. The proof is by an averaging argument, and exploits the fact that a and b are not adjacent. Let us consider the following two vertex sets $A = \{a, b\}$, and $B = T - L - A$. In each step of a construction of T from $T - L$, we show that the average degree of vertices in B is more than the average degree of vertices in A. Clearly, in $T-L$, the average degree in A is k, and in B it is $k+1$. Whenever a new vertex is added, it must be adjacent to at least $k-1$ vertices in B and at most one vertex in A. Therefore, after adding m vertices, the average degree of A will be at most $k+\frac{m}{2}$, and the average degree of vertices in B will be at least $k+1+\frac{m(k-1)}{k}$. So the degrees of vertices in $T-L$ can never become all equal. Therefore, $d_1 = \ldots = d_{n-k-2} = k, d_{n-k-1} = \ldots = d_n = b$ is not the degree sequence of a k-tree. □

3 Integer Sequences That Characterize *k*-Trees

Definition 4. *The* $(k, k + 1)$-degree *of a k-clique C in a graph G is defined as the number of $(k + 1)$-cliques in G which contain C. The* $(k, k + 1)$-degree sequence *of a graph G is the list of $(k, k + 1)$-degrees of the k-cliques in G, with duplicates, sorted in non-decreasing order.*

The $(1, 2)$-degree sequence of a graph is its degree sequence, and its $(2, 3)$-degree sequence can be thought of as the *edge-triangle* degree sequence.

Definition 5. *For $n \geq k + 1$, a sequence of integers $\Delta =< d_1, d_2, \ldots, d_r >$ is a* $(k, k + 1)$-sequence *if the following conditions hold:*

1. $r = k + 1 + (n - k - 1)k$.
2. $d_i \leq d_{i+1}$ *for all* $1 \leq i < r$.
3. *If* $n = k + 1$, $d_i = 1$ *for* $1 \leq i \leq k + 1$. *If* $n > k + 1$, *then* $d_i = 1$ *for* $1 \leq i \leq 2k$.
4. $\sum_{i=1}^{r} d_i = (k + 1)(n - k)$.

The following two lemma follow from the definition of a $(k, k+1)$-sequence, and are used in the proof of Theorem 3, which is our main theorem.

Lemma 2. *For $n = k + 1$, the $(k, k + 1)$-sequence is unique and every element is a 1. For $n = k + 2$, the $(k, k + 1)$-sequence is unique; $d_1 = d_2 = \ldots = d_{2k} = 1, d_{2k+1} = 2$.*

Lemma 3. *Let $r > k + 1$. Let $< d_1, \ldots, d_r >$ be a $(k, k + 1)$-sequence and let l be the smallest integer such that $d_l > 1$. If $< d_{k+1}, \ldots, d_l - 1, \ldots, d_r >$ is the $(k, k+1)$-degree sequence of a k-tree, then $< d_1, \ldots, d_r >$ is the $(k, k+1)$-degree sequence of a k-tree.*

Theorem 2. *Let $\Delta =< d_1, d_2, \ldots, d_r >$ be a sequence of integers. Then Δ is the $(k, k + 1)$-degree sequence of a k-tree iff Δ is a $(k, k + 1)$-sequence.*

*Proof.*First we prove the necessary condition. In a k-tree on n vertices, the number of k-cliques, denoted by r, is $k(n-k)+1$. Further, the sum of the entries in the $(k, k+1)$-degree sequence $< d_1, d_2, \ldots, d_r >$ is $\sum_{i=1}^{r} d_i = (k+1)(n-k)$. The proofs of these claims are by induction on n. The base case is for $n = k+1$; in this case there are k k-cliques, a unique $k+1$-clique, and the sum of the degrees is $k+1$. To complete the induction, if we assume that these formulas hold for n, proving that they hold for $n+1$ follows by simple arithmetic. We now prove the property on the entries of the degree sequence. If $n = k+1$, as observed before, there are k k-cliques, and a unique $k+1$-clique. So the degree sequence is $d_1 = d_2 = \ldots = d_{k+1} = 1$. For the case of $n > k+1$, we observe a simple invariant maintained in every k-tree: there are two vertices of degree k, this property is easily seen in the inductive construction of k-trees. Further, in a k-tree a vertex of degree k is present in exactly k k-cliques. Each of these k-cliques is contained in the unique $k+1$-clique induced by the vertex and all its neighbors. The entries corresponding to these k k-cliques are 1 in the $(k, k+1)$-degree sequence. Since there are two vertices of degree k in any k-tree, it follows that there are $2k$ 1's in the $(k, k+1)$-degree sequence. Therefore, $d_1 = d_2 = \ldots = d_{2k} = 1$.

We prove the sufficient condition by induction on the length of the $(k, k+1)$-sequence. Let us consider a $(k, k+1)$-sequence $d_1, d_2, \ldots, d_r$. If $r = k+1$, then the corresponding k-tree is the clique of $k+1$ vertices. If $r = 2k+1$, then the corresponding k-tree has $k+2$ vertices in which there is a k-clique, and two non-adjacent vertices are both adjacent to each vertex in the k-clique. There are no other vertices and edges in the graph. Therefore, $(k, k+1)$-sequences of length $k+1$ and $2k+1$ can be realized by k-trees, which is the base case for our induction. Let us consider the case when $r > 2k+1$. Let l be the smallest integer such that $d_l > 1$. Clearly, $l > 2k$. We show that $d_{k+1}, \ldots, d_l - 1, \ldots, d_r$ is a $(k, k+1)$-sequence. The sum of the degrees is clearly $(k+1)(n-1-k)$. We only need to show that $d_{k+1} = d_{k+2} = \ldots = d_{3k} = 1$. If we assume not that is we assume that $d_{k+1} = \ldots d_b = 1, b < 3k$. Then it follows that we have $k(n-1-k)+1-b+k$ entries in the sequence which are more than 1. Further, we also know that the sum of these entries is $(k+1)(n-1-k)-b+k$. It now follows that $(k+1)(n-1-k)-b+k \geq 2k(n-1-k)+2-2b+2k$, that is $b \geq (n-1-k)(k-1)+k+2$. If $b < 3k$, then it follows that $2k-2 > (n-1-k)(k-1)$, which in turn implies that $n < 2+(k+1)$, that is $n = k+1$ or $n = k+2$. This means $r \leq 2k+1$, a contradiction to the fact that we are considering $r > 2k+1$. Therefore our assumption that $d_{k+1}, \ldots, d_r$ is not a $(k, k+1)$-sequence is wrong. Inductively, $d_{k+1}, \ldots, d_l - 1, \ldots, d_r$ is the $(k, k+1)$-degree sequence of a k-tree. By Lemma 3 it now follows that $d_1, \ldots, d_r$ is also the $(k, k+1)$-degree sequence of a k-tree. Hence the characterization is complete. □

4 Sufficient Conditions for 2-Trees

In this section we present our main results on the sufficient conditions on the degree sequence of 2-trees.

We call a 2-tree, which contains exactly two leaves, a 2-*chain*. In a 2-tree T, a *pruning sequence* is a minimal sequence of degree 2 nodes of T such that after removing these nodes according to the sequence, we get a 2-chain. The process of applying a pruning sequence to a 2-tree is called pruning.

Definition 6. *A* 2-caterpillar *is either a 3-clique, or a* 2*-tree with a pruning sequence.*

Definition 7. *For each* $l \geq 1$*, a* $[d_1, d_2, \ldots, d_l]$-path *is a path* $v_1, \ldots, v_l$ *such that for* $1 \leq i \leq l$*, the degree of* v_i *is* d_i*. For* $l = 2$*, we refer to a* $[d_1, d_2]$*-path as a* $[d_1, d_2]$-edge.

Theorem 3. *If a plausible* 2*-sequence contains at least one* 3*, then it is the degree sequence of a* 2*-tree. Furthermore, if* $n > 4$ *then there is* 2*-tree realizing this degree sequence in which there is a* $[2, 3, min_1]$*-path, where* $min_1 = d_l$ *and* $d_l \geq 4$ *but* $d_{l-1} < 4$*. If* $l < n$*, then there is even a* $[2, 3, min_1, min_2]$*-path. Here* $min_2 = d_{l+1}$*, i.e.,* min_2 *is the next degree in the sequence.*

The proof of this theorem will be by induction on n, i.e., the number of vertices. In the induction step, certain boundary cases can occur. These special cases are dealt with by the following lemmas.

Lemma 4. *If in a plausible* 2*-sequence* $d_{n-1} < 4$*, then the sequence contains exactly two* 2*'s and* $d_n = n - 1$*. Further, such a sequence is the degree sequence of a* 2*-tree. In this special case, Theorem 3 holds.*

Proof. Since $\sum_{i=1}^{n} d_i = 4n - 6$, and the fact that $d_1 = d_2 = 2$, it follows that $\sum_{i=3}^{n} d_i = 4n - 10$. Hence, $\sum_{i=3}^{n-1} d_i \geq 3n - 9 = 3(n-3)$ as $d_n \leq n-1$. Therefore, the average value of $\{d_3, \ldots, d_{n-1}\}$ is at least 3. Since $d_{n-1} < 4$ it follows that $\sum_{i=1}^{n-1} d_i = 2t + 3(n - t - 1) = 3n - t - 3$, where t is the number of 2's in $d_1, \ldots, d_{n-1}$. Therefore, $d_n = n + t - 3$. Since $d_n \leq n - 1$, it follows that $t \leq 2$. Therefore, $t = 2$, and consequently, $d_3 = \ldots = d_{n-1} = 3, d_n = n - 1$. This sequence is trivially realized by a "fan": a central node of degree $n - 1$, which is surrounded by nodes of degree 3 with a node of degree 2 at either end of this ring. □

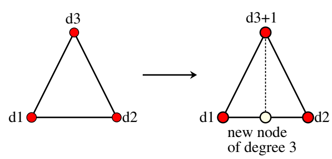

Fig. 1. Inserting a node of degree 3 to a $[d_1, d_2, d_3]$-triangle and changing the degree of only one node from d_3 to $d_3 + 1$

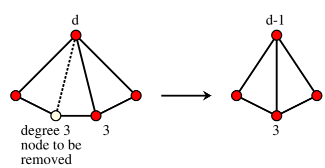

Fig. 2. Deleting a node of degree 3 from a $[3, 3, d]$-triangle and changing the degree of only one node from d to $d - 1$

Lemma 5. *A plausible* 2*-sequence with exactly two* 2*'s in the sequence will also contain at least one* 3*. Such a sequence is the degree sequence of a* 2*-tree. In this special case, Theorem 3 holds. In fact, the nodes of high degrees* $h_1, h_2, \ldots, h_r$*, where the* h_j *are the subset of the degrees* d_i *with* $d_i \geq 4$*, can be arranged in any arbitrary order such that there is a* $[2, 3, h_{j_1}, \ldots, h_{j_r}]$ *path.*

Proof. In the explicit construction we will, first, reduce all nodes of degree > 4 to degree 4 and then remove the "appropriate" number of 3's from the sequence, namely, a node of degree $4+x$ corresponds to x nodes of degree 3, as the sum is fixed at $4n-6$ and there are only two 2's. This, in the end, leaves a sequence of the form $2, 2, 3, 3, 4, 4, \ldots, 4$ with exactly two 2's, two 3's and the same number of 4's as nodes of high degree in the original sequence. This sequence is then realizable by a "straight chain". See Figure 3 for an illustration. Once we have this basic backbone, we fix a $[2, 3, 4, 4, 4, \ldots, 4]$-path and identify the 4's with the desired degrees $h_{j_1}, \ldots, h_{j_r}$. For each such node of intended degree h_{j_s} we then insert $(h_{j_s} - 4)$ 3's into the 2-tree, as illustrated in Figure 1. Figure 3 illustrates the whole process. □

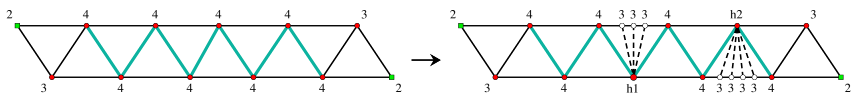

Fig. 3. Inserting $(h_i - 4)$ degree 3 nodes on a $[4, \ldots, 4]$-path (shown in bold), changing the degree of degree 4 nodes to h_i, if $h_i > 4$. The nodes are labelled by their degrees.

Lemma 6. *If* $d_n = n-1$ *in a plausible* 2*-sequence, then the sequence is the degree sequence of a* 2*-tree. In fact, the nodes of high degrees* $h_1, h_2, \ldots, h_r$*, where the* h_j *are the subset of the degrees* d_i *where* $d_i \geq 4$*, can be arranged in any arbitrary order such that, if the* 2*-sequence contains a* 3*, there is a* $[2, 3, \ldots, 3, h_{j_1}, \ldots, h_{j_r}]$ *path passing through all nodes of degree* 3 *consecutively or, if the* 2*-sequence does not contain a* 3*, there is* $[2, h_{j_1}, \ldots, h_{j_r}]$ *path.*

Proof. Note that the combination of the conditions of Lemmas 5 and 6 leads to the very strict conditions of Lemma 4. So we can assume that there are at least three 2's in the sequence. The proof of the lemma is by induction. The induction starts at $n = 5$ with the only plausible sequences $< 2, 2, 2, 4, 4 >$ and $< 2, 2, 3, 3, 4 >$, both of which are realizable as desired by inspection. Now suppose the lemma holds for up to n. Note that we can always assume that h_{j_1} is not the maximum degree $n-1$ (for n nodes) as, if the sequence is realizable, the node of maximum degree will be connected to *all* other nodes and can thus be inserted anywhere along an existing path. So, we can first move it to "the end" by assuming $h_{j_r} = n-1$. If there is only one node of high degree ≥ 4, then we are also in the case of Lemma 4. Now, given a 2-sequence with $n+1$ degrees and $d_{n+1} = n$ and $4 \leq h_{j_1} < n$, simply remove a 2, as there are at least three 2's by the comment before, and reduce both the maximum degree d_{n+1} and the degree h_{j_1} by one and apply induction. As the node of maximum degree, which now has degree $n-1$, is still connected to all remaining nodes it is, in particular, connected to the node of degree $h_{j_1} - 1$. Hence, we can put a leaf back on top of the $[n-1, h_{j_1} - 1]$-edge to get back the original degree sequence. □

The following observation, illustrated in Figure 2, will allow us to reduce the sequences of 3's along the path to a single 3.

Observation 1. *Given a $[3,3]$-edge as part of a $[3,3,d]$-triangle in a 2-tree, we can remove one of the two 3's while also reducing d to $d-1$ and we obtain another 2-tree.*

Corollary 1. *If $d_n = n-1$ in a plausible 2-sequence, then the sequence is the degree sequence of a 2-tree. In fact, if the 2-sequence contains at least one 3, then the nodes of high degrees $h_1, h_2, \ldots, h_r$, where the h_j are the subset of the degrees d_i with $d_i \geq 4$, can be arranged in any arbitrary order such that there is a $[2, 3, h_{j_1}, \ldots, h_{j_r}]$ path. Thus, in particular, for $d_n = n-1$ Theorem 3 holds.*

Proof. Note that if, in the case where $d_n = n-1$, we have a $[3,3]$-edge, then both corresponding nodes must also be connected to the central node of degree $n-1$. Thus, using the observation above, we can remove one of the nodes of degree 3 along with its edge connected to the node of degree $n-1$ (now becoming $n-2$) and bridge its other two edges thereby leaving all other degrees unchanged. If we now put a leaf on top of the other leaf, which is not involved in the desired path, then this 2 becomes a 3, we insert another 2 (the newly added leaf) and the central node of degree $n-2$ goes back to degree $n-1$. Using this trick repeatedly, we can remove any sequence of 3's along the path to a single 3. □

With these lemmas, we can now prove the Theorem 3.

Proof. The statement is proved constructively by induction on n. The statement holds for $n = 4$. At each step, if we ever get left with (a) only one high degree greater than 3, (b) only two 2's or (c) $d_n = n-1$, then we refer to the lemmas above, namely Lemma 4, Lemma 5 and Corollary 1

Case 1. Assume that we have no 4's in the sequence, so $\min_1 > 4$ and $\min_2 > 4$. Then reduce $\min_1$ and $\min_2$ by 1 and remove a 2 from the sequence. This gives another plausible 2-sequence and there will also remain at least one 3. So, by the induction hypothesis, construct a 2-tree with a $[2, 3, \min_1 - 1, \min_2 - 1]$-path. Observe that by reducing the two minima among the vertices of degree more than 3, they will still remain the minima, as $\min_1, \min_2 > 4$. Add a vertex to this 2-tree and connect it to the two last nodes on this path. This gives a 2-tree realizing our original sequence of length n.

Case 2. Now assume we have at least one 4 in the sequence, so $\min_1 = 4$. Then reduce a 3 to a 2, reduce a 4 to a 3 and remove a 2. Again, this will give a plausible 2-sequence of shorter length with at least one 3. Observe that $\min_2$ has now become the smallest high degree. By induction, we then get a $[2, 3, \min_2, x]$-path for some x. Add a vertex and connect it to the first two nodes on this path. This then gives a $[2, 3, 4, \min_2]$-path. □

5 Almost Every Plausible 2-Sequence Is Realizable

Let the partition function $p(n)$ give the number of ways for writing a positive integer n as a sum of positive integers, where the order of addends is not considered. From [10,11,17], we know the following asymptotic formula.

Theorem 4. *As $n \to \infty$, $p(n) \to \frac{\exp(\pi\sqrt{2n/3})}{4\sqrt{3}n}$.*

Lemma 7. *The number of plausible 2-sequences of size n is at most $p(2n-6)$.*

Proof. The lemma follows from the fact that every plausible 2-sequence Δ of size n defines a unique partition of the number $2n-6$. It is because, by subtracting 2 from each number of Δ, we get a monotonic sequence of n non-negative numbers, whose sum is $(4n-6)-2n = 2n-6$. □

Lemma 8. *The number of plausible 2-sequences of size n containing at least one 3 is greater than $p(2n-7) - 2n \cdot p(n)$.*

Proof. Let $\Delta =< d_i >_{i=1}^n$ be a plausible 2-sequence of size n containing at least one 3. Since the sum of all d_i's is $4n-6$ and since Δ contains at least two 2's and one 3, the sum of the remaining $n-3$ elements in Δ is $4n-13$. Now, since all the elements are bigger than or equal to 2, Δ defines a partition of the number $(4n-13)-2(n-3) = 2n-7$ into $n-3$ blocks. However, not all partitions $(b_1, \ldots, b_l)$ of $2n-7$ correspond to plausible 2-sequences. There are two types of partitions which do not correspond to plausible sequences. First, the partition may contain more than $n-3$ blocks and thus cannot correspond to a 2-sequence of size n; we call such a partition a "long partition". Below we show that the total number of such partitions is bounded from above by $np(n)$. Since the order of the partition is not considered, we can assume that a partition is sorted non-increasing order. Therefore, if the partition is "long", then $b_{n-2} = 1$, because $b_i \geq 1$ for all $i \leq n-3$ and $\sum_{i=1}^{n-2} b_i \leq 2n-7$. Therefore, $b_{n-2+i} \leq 1$ for all $i \geq 1$ and there is a unique j, determined by the sum of $b_1, \ldots, b_{n-3}$ such that $b_l = 0$ for $l \geq j$. Hence, a "long" partition is determined by the sum S of the first $n-3$ elements of the partition. Since $n-5 = 2n-7-(n-2)$, it follows that $n-3 \leq S \leq 2n-8$ and therefore the number of long partitions is exactly $\sum_{i=1}^{n-5} p(i)$. Finally, since $p(n) > p(n-5-i)$ for all $i = 1, 2, \ldots, n-5$, it follows that the number of long partitions is less than $np(n)$.

The other type of partitions of $2n-7$ that do not correspond to plausible k-sequences are those for which the biggest number is greater than $n-3$, leading to a 2-sequence which violates the maximum condition. The sum of the rest of the numbers in the partition is between 1 and $n-5$. Therefore the number of such partitions is at most $\sum_{i=1}^{n-5} p(i)$, which is again less than $np(n)$. Thus the lemma follows. □

Theorem 5. *Almost every plausible 2-sequence is realizable by a 2-tree.*

Proof. By Theorem 3, it is enough to show that almost every plausible 2-sequence contains a 3. Consider a random experiment that picks a sequence randomly from the set of all plausible 2-sequences. Denote by A an event that the picked sequence contains a 3. By Lemma 7 and 8, we have $\Pr[A] \geq \frac{p(2n-7)-2n \cdot p(n)}{p(2n-6)}$. From Theorem 4 we know that the right hand side approaches 1 as n approaches ∞, so we have $\lim_{n\to\infty} \Pr[A] = 1$, hence the result. □

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